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PARAMETRIC RESONANCE OF VISCOELASTIC COLUMNS†

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Abstract—The results of an analytical and experimental investigation of the effect of viscoelastic material behavior on the instability regions of a column subjected to a periodic axial load $P_0 + P_1 \cos \omega t$ are presented. The complex modulus representation is used for the material. The effect of various types of material response is investigated. Viscoelastic effects are found to have a pronounced influence on the instability regions. The instability regions of a polymethyl methacrylate (Plexiglass) column are determined experimentally, and the data are found to be in reasonable agreement with the theory.

INTRODUCTION

IF A straight column is subjected to a periodic axial load it generally will undergo forced longitudinal vibration. However, this motion becomes unstable over certain regions of the load amplitude–frequency parameter space (instability regions), and a lateral vibration results. This type of response is usually referred to as parametric resonance or parametric instability. In the simplest case, this behavior is governed by Mathieu's equation, and the stability characteristics of the column are given by the Strutt diagram.

In previous investigations of the parametric response of structures, the material properties generally were assumed to be rate (frequency) independent; see [1, 2] for a list of references. It would appear that the results obtained in these investigations are not applicable to viscoelastic structures, in as much as viscoelastic materials exhibit significant rate effects. Since viscoelastic material behavior is increasingly common in practical structural problems, its effect on the parametric response is of general interest.

The problem of the parametric response of viscoelastic structures has received little attention to date. Kovalenko [3] has considered the problem of a column of constant stiffness with internal damping linearly proportional to the strain rate. Stevens [4] also considered the column problem, assuming various simple spring–dashpot representations for the material. Both of these investigations were restricted to the determination of the instability regions. It was found that the shape, size, and distribution of these regions in the parameter space were significantly influenced by viscoelastic effects. However, these results must be regarded as purely qualitative because of the nature of the assumptions made about the material behavior.

In this paper we present the results of an analytical and experimental investigation of the effect of viscoelastic material behavior on the instability regions of a column. The

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analysis is valid for material properties which are arbitrary functions of frequency, and is based on the complex modulus representation for viscoelastic materials. This representation is valid only for linear materials. Many polymeric materials meet this requirement, particularly at the low stress levels typically encountered in columns. On the other hand, the behavior of metals (and of some nonmetallic structural materials) is often highly nonlinear. The analysis presented may not be realistic for these materials.

ANALYTICAL WORK

The structure considered is a straight, simply supported column of uniform cross section subjected to an axial load $P_0 + P_1 \cos \omega t$ as shown in Fig. 1. The length of the column will be denoted by L, the mass per unit length by m, and the moment of inertia of the cross section by I.

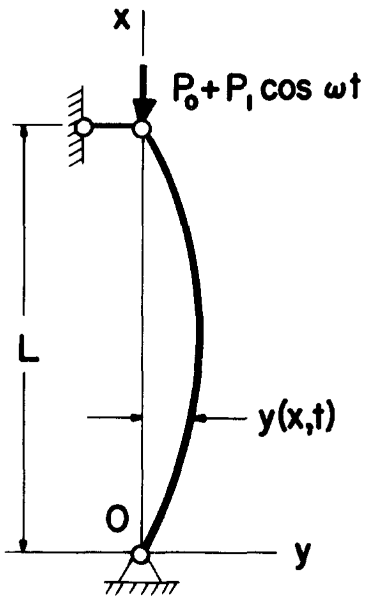

FIG. 1. Column configuration.

The column material is assumed to be linearly viscoelastic. Consequently, the material properties can be represented by the complex modulus $E_1 + iE_2$, where E_1 determines the material stiffness and E_2 determines the damping capacity; see [5, 6]. In addition to providing an accurate representation of the material properties over the entire frequency range of interest, this approach leads to tractable equations. It has the added advantage that there are standard experimental techniques for determining the material properties. In addition, the complex modulus notation covers the consideration of other types of material behavior as special cases. For example, the case of hysteretic (rate independent) damping is obtained by letting E_1 and E_2 be constants [7]. By letting E_1 be constant and E_2 be proportional to the frequency, ω, we get the case of an elastic material with external viscous damping [8]. Of course, the case where E_1 is constant and E_2 is zero corresponds to purely elastic material behavior.

In the analysis which follows, it will be assumed that the column is subjected to a constant, uniform temperature. In this case the complex modulus is a function of frequency only. Heating effects due to energy dissipation in the material are not considered. It is believed that these effects will be small for the stress levels and frequencies encountered in the current problem.

Using simple beam theory and neglecting axial inertia, the governing equation of motion is found to be

$$I(E_1+iE_2)\frac{\partial^4 y}{\partial x^4}+(P_0+P_1\cos\omega t)\frac{\partial^2 y}{\partial x^2}+m\frac{\partial^2 y}{\partial t^2}=0, \tag{1}$$

subject to the boundary conditions

$$y(0,t)=y(L,t)=\frac{\partial^2 y}{\partial x^2}(0,t)=\frac{\partial^2 y}{\partial x^2}(L,t)=0.$$

An approximate solution to (1) is sought in the form

$$y(x,t)=A(t)\sin(\pi x/L). \tag{2†}$$

Substituting (2) into (1), we obtain

$$\frac{\mathrm{d}^2 A}{\mathrm{d}t^2}+\Omega^2(1+i\alpha-2\mu\cos\omega t)A=0 \tag{3}$$

where

$$\begin{aligned}\Omega^2 &= (\pi^4 E_1 I/mL^4)(1-P_0/P^*),\\ P^* &= \pi^2 E_1 I/L^2,\\ 2\mu &= P_1/(P^*-P_0),\\ \alpha &= (E_2/E_1)/(1-P_0/P^*).\end{aligned}$$

Physically, Ω and P^* represent the natural frequency of lateral vibration and the Euler critical load, respectively, of an elastic column with modulus E_1.

Equation (3) is a Mathieu equation with complex coefficients. The theory of Floquet still applies in this case [10]. Consequently, the boundaries of the instability regions correspond to periodic solutions of (3) with period $2\pi/\omega$ and $4\pi/\omega$. When damping is present, usually only the first instability region is of any major significance; the higher order regions correspond to values of the load parameter μ larger than those encountered in most practical problems [1]. Hence in the following we consider only the first (principal) instability region. This region corresponds to a periodic solution of (3) with period $4\pi/\omega$.

† The general solution is

$$y(x,t)=\sum_{n=1}^{\infty} A_n(t)\sin(n\pi x/L).$$

However, previous experiments on elastic columns [9] and the experiments described herein show that the first spatial mode ($n=1$) is predominant.

The procedure used for determining the boundary of the instability region is a conventional one; see [1]. A periodic solution of (3) with period $4\pi/\omega$ is sought in the form

$$A(t) = \sum_{k=1,3,5}^{\infty} a_k \sin(k\,\omega t/2) + b_k \cos(k\,\omega t/2). \tag{4}$$

We substitute (4) into (3), carry out the trigonometric transformations, and equate coefficients of like $\sin(k\,\omega t/2)$ and $\cos(k\,\omega t/2)$ terms. This yields a system of linear homogeneous algebraic equations for the coefficients a_k and b_k. In the present case it is expedient to use complex notation in carrying out the steps indicated.

Because of the frequency dependence of the material properties there is a complication here which does not arise in the elastic case. The parameters (Ω, α, μ) in (3) take on different values at each of the different frequencies in the bending response, and care must be taken to associate the correct values of these parameters with each of the terms in (4). For example, all parameters associated with the kth terms in (4) should be evaluated at the frequency $k\omega/2$. Now the question may arise as to whether the parameters in (3) should be evaluated at the bending frequencies as indicated, or at the frequency of the applied load. This is a possible source of confusion since the two frequencies are generally not the same. One must keep in mind that the parameters in (3) are associated with bending, and hence should be evaluated at the bending frequencies. This conclusion is confirmed by the experimental results and by the results obtained in [4].

Applying the procedure outlined, we obtain the following system of equations for the coefficients a_k and b_k.

$$\begin{aligned}
&(1-n_1^2-\mu_1)b_1+\alpha_1 a_1-\mu_3(\Omega_3/\Omega_1)^2 b_3 = 0,\\
&(1-k^2n_k^2)b_k+\alpha_k a_k-\mu_{k-2}(\Omega_{k-2}/\Omega_k)^2 b_{k-2}-\mu_{k+2}(\Omega_{k+2}/\Omega_k)^2 b_{k+2} = 0 \qquad k = 3,5,7\ldots\\
&(1-n_1^2+\mu_1)a_1-\alpha_1 b_1-\mu_3(\Omega_3/\Omega_1)^2 a_3 = 0, \qquad (5)\dagger\\
&(1-k^2n_k^2)a_k-\alpha_k b_k-\mu_{k-2}(\Omega_{k-2}/\Omega_k)^2 a_{k-2}-\mu_{k+2}(\Omega_{k+2}/\Omega_k)^2 a_{k+2} = 0 \qquad k = 3,5,7\ldots
\end{aligned}$$

Here $(\Omega_k, \alpha_k, \mu_k)$ denote (Ω, α, μ) evaluated at $k\omega/2$ and

$$n_k = \omega/2\Omega_k.$$

The requirement that the determinant of the coefficients of a_k and b_k be equal to zero leads to a relationship between n_k, μ_k and α_k which defines the boundary of the instability region in an (n_k, μ_k, α_k) parameter space.‡

An nth order approximation to the instability region is obtained by retaining the first n terms in (4). Usually the first or second approximation gives sufficient accuracy if μ is not too large. The resulting expressions for the first and second approximations to the boundary of the instability region are, respectively.

$$\mu_1^2-\alpha_1^2-(1-n_1^2) = 0 \tag{6}$$

† For $\alpha = 0$ and α proportional to ω, (5) reduces to the corresponding equations for an elastic column and an elastic column with external viscous damping, respectively. This is readily confirmed by making the appropriate substitutions and comparing the results with the equations given by Bolotin [1].

‡ The resulting determinant is infinite and its convergence must be considered. However, it is easily shown that this determinant can be reduced to a normal determinant, which is known to be convergent [11].

and

$$
\begin{aligned}
&(\mu_1\mu_3)^2+2\mu_1\mu_3[\alpha_1\alpha_3-(1-n_1^2)(1-9n_3^2)]-\mu_1^2[(1-9n_3^2)^2+\alpha_3^2] \\
&\quad +[(1-n_1^2)^2(1-9n_3^2)^2+\alpha_1^2(1-9n_3^2)^2+\alpha_3^2(1-n_1^2)^2+(\alpha_1\alpha_3)^2] = 0.
\end{aligned}
\tag{7}
$$

In order to display fully the effect of the frequency dependence of the material properties, it is necessary to base all calculations on some reference value of the modulus. To this end, we introduce the quantities

$$
\begin{aligned}
\bar{E} &= E_1/E_r, \\
\Omega_r^2 &= (\pi^4 E_r I/mL^4)(1-P_0/P_r^*), \\
P_r^* &= \pi^2 E_r I/L^2, \\
2\mu_r &= P_1/(P_r^*-P_0), \\
n_r &= \omega/2\Omega_r,
\end{aligned}
\tag{8}
$$

where E_r is a reference value of E_1. Stated in another way, we introduce a fictitious elastic column with modulus E_r which serves as a basis of comparison for all results. The choice of E_r is arbitrary, but it is probably most meaningful to take the static modulus as the reference value. Introducing the quantities in (8) into (6) and (7), we obtain, respectively,

$$
\mu_r^2-(G_1-n_r^2)^2-\alpha_1^2 G_1^2 = 0 \tag{9}
$$

and

$$
\begin{aligned}
&\mu_r^4+\mu_r^2[2(\alpha_1 G_1)(\alpha_3 G_3)-(\alpha_3 G_3)^2-2(G_1-n_r^2)(G_3-9n_r^2)-(G_3-9n_r^2)^2] \\
&\quad +[(\alpha_1 G_1)^2(\alpha_2 G_2)^2+(\alpha_1 G_1)^2(G_3-9n_r^2)^2+(\alpha_3 G_3)^2(G_1-n_r^2)^2+(G_1-n_r^2)^2(G_3-9n_r^2)^2] = 0
\end{aligned}
\tag{10}
$$

where

$$
G_k = \frac{(\bar{E}_k - P_0/P_r^*)}{(1-P_0/P_r^*)}
$$

and $\bar{E}_k$ denotes $\bar{E}$ evaluated at $k\omega/2$. The effect of the system parameters can now be effectively displayed by plotting μ_r vs. n_r for various values of the other parameters.

If the variation of the complex modulus with frequency is known, the instability region can be determined from (9) or (10). Material data in the form of experimental curves can be used, but data in the form of empirical equations is preferable. This simplifies the numerical work and makes it easier to determine the effect of the various material parameters. As we shall see later, the instability region encompasses a relatively narrow range of frequencies. Hence, in the frequency intervals of interest, the material properties can usually be represented quite accurately by emperical equations of the form

$$
\begin{aligned}
E_1 &= a\omega^p \\
E_2 &= b\omega^q
\end{aligned}
\tag{11}
$$

where a, b, p, and q are constants. In these equations the independent variable ω is not to be confused with the frequency of the applied load.

From (9) we see that the minimum value of the load parameter μ_r for which parametric instability can occur is

$$\mu_{r_{min}} = \alpha_1 G_1$$

or expressed in terms of (11)

$$\mu_{r_{min}} = \frac{(b/E_r)(\Omega_r n_{r_{min}})^q}{(1-P_0/P_r^*)}. \tag{12}$$

In (12), $n_{r_{min}}$ is the value of the frequency parameter n_r corresponding to $\mu_{r_{min}}$; it is given by the transcendental equation

$$(1-P_0/P_r^*)n_{r_{min}}^2 - (a/E_r)(\Omega_r n_{r_{min}})^p + (P_0/P_r^*) = 0. \tag{13}$$

EXPERIMENTAL WORK

The experimental portion of the investigation consisted of two parts: one part to determine the material properties, and the second part to check the validity of the theory. The column and material test specimens were fabricated from a commercially available polymethyl methacrylate (Plexiglass). All experiments were conducted at room conditions.

The vibrating reed method [12] was used to determine the complex modulus over the frequency range 15 to 1000 Hz (100 to 6000 rad/sec). It was found that, in this range, the data could be represented by equation (11) to a high degree of accuracy. The constants in equation (11), a, b, p, and q, were determined by fitting the experimental data to a straight line on a log–log scale using a least squares procedure.† The static modulus of elasticity, which was used as the reference modulus in all calculations, was determined from compression tests.

Experiments were then conducted to determine the instability region of the column. A full description of the column specimen is given below:

Material		polymethyl methacrylate (Plexiglass)
Static Modulus		$4{\cdot}6 \times 10^5$ psi
a	$100 \text{ rad/sec} < \omega < 6000 \text{ rad/sec}$	$4{\cdot}38 \times 10^5$ psi
b		$7{\cdot}28 \times 10^4$ psi
p		0·0815
q		−0·0635
Length		17·53 in.
Cross Section		rectangular 1·010 in. × 0·236 in.
P_r^* (Euler critical load)		16·4 lb
Ω_r (Natural frequency of lateral vibration)		$141(1-P_0/P_r^*)^{1/2}$ rad/sec

It is of interest to note that the real part of the modulus increases with increasing frequency ($p > 0$), while the imaginary part decreases ($q < 0$). The Euler critical load, P_r^*, was determined experimentally using the method of Southwell [13]. The value obtained was in excellent agreement with the theoretical value, thus indicating that the experimental setup for the column provided simply supported end conditions.

† A certain amount of scatter in the data is unavoidable. It is important that the material representation be the "best possible," since the instability region is quite sensitive to the material properties—the damping in particular. Considerable care must also be exercised in performing the complex modulus experiments.

The experimental apparatus for the column tests was essentially the same as that used by Somerset and Evan-Iwanowski in previous experiments on elastic columns [9]. The reader is referred to their paper for details.

A schematic of the experimental setup is shown in Fig. 2. The static component of load, P_o, is provided by an initial compression in the coupling spring. The dynamic component,

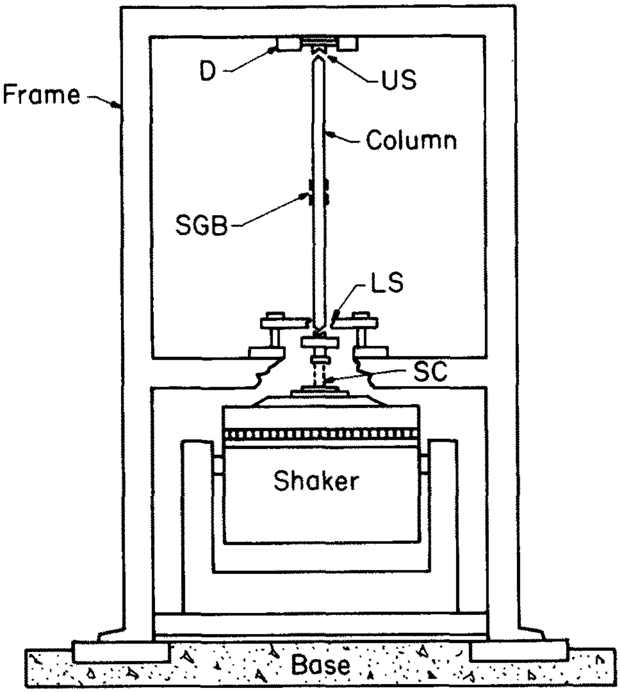

FIG. 2. Column mounting and loading system. D—dynamometer; US, LS—upper, lower support; SC—spring connector; SGB—strain gage bridge.

$P_1 \cos \omega t$, is supplied by the electro-magnetic shaker via the coupling spring. The axial load is measured by means of the load cell, and the lateral response of the column is monitored by electrical resistance strain gages. The axial load and lateral deflection are simultaneously recorded on a continuous strip recorder. The ends of the column are beveled and are placed in notched plates at each end to produce simply supported end conditions.

The experimental procedure was as follows: The column was mounted in the apparatus, and the desired static load applied by compressing the coupling spring by means of an adjusting nut. The frequency control of the shaker was then set at the desired loading frequency, and the amplitude of the dynamic load, P_1, slowly increased until the column exhibited parametric vibrations. In the present setup, this was detected visually from the recorder traces. The value of P_1 and ω corresponding to the onset of the parametric vibrations locates one point on the boundary of the instability region. The frequency was then changed, and the process repeated until the complete instability region had been traced out. The results obtained are presented and discussed in the following section. In every instance the column vibrated in the first spatial mode when it became parametrically unstable.

RESULTS AND CONCLUSIONS

Numerous computations were made in order to assess the effect of the different material parameters on the instability region. The results shown in Figs. 3–5 are typical of those obtained. Except where noted, all numerical results presented are for the column specimen described in the preceding section, and are based on the second approximation to the instability region as given by (10). The static modulus of elasticity is used as the reference modulus throughout.

The numerical results indicate the viscoelastic effects have a significant influence on the instability region. This is illustrated in Fig. 3, where the instability region for the viscoelastic column is compared with the one obtained by assuming the material is elastic with modulus E_r. The instability region for the viscoelastic column is similar in appearance to that of an elastic column with external viscous damping [1, 9]. However, the shift in the n_r direction is much more pronounced in the present case because of the frequency dependence of the material stiffness. The instability regions based on the first approximation, (9), are also shown. As can be seen, the first approximation gives good results, and is sufficiently accurate for most purposes.

Figures 4 and 5 show the effect of the material parameters p and q on the instability region. Recall that p and q are measures of the rate of change with frequency of the material stiffness and damping, respectively. The figures show that q affects only the position of the instability region in the μ_r direction, while p affects only the position in the n_r direction. That this is the case is readily evident from (12) and (13), where it is seen that $n_{r_{min}}$ depends on p only, while $\mu_{r_{min}}$ depends on q only.

In Fig. 6, the theoretical instability region for the polymethyl methacrylate (Plexiglass) column is compared with the experimental results for various values of P_o. The agreement between theory and experiment is reasonably good. The fact that the experimental results correspond to higher values of μ_r than do the analytical results is due, at least in part, to the experimental procedure. The boundary of the theoretical region corresponds to the onset of parametric vibrations. In the experiments it was difficult to determine just when this took place, since the start of the parametric vibration was obscured by a forced vibration arising from a slight initial curvature of the column. The parametric vibration was observable only when its amplitude exceeded that of the forced vibration. The experimental values of μ_r thus correspond to parametric vibrations of finite amplitude, rather than to the onset of parametric vibrations. However, preliminary calculations indicate that this effect is not large enough to account completely for the discrepancy between theory and experiment.

It is likely that this discrepancy is due to a number of factors, in as much as any factor which gives rise to more damping (real or apparent) than included in the analysis will cause the experimental instability region to lie above the theoretical one. Examples which immediately come to mind are damping in the supports and damping due to the surrounding medium (air in the present case). Material nonlinearities and heating effects are other possibilities, although preliminary calculations based on existing material data indicate that they have a relatively small effect on the instability region for the conditions encountered in the experiments reported here.

The alignment of the theoretical and experimental instability regions in the n_r direction is seen to be very good. This indicates that the material stiffness was accurately predicted.

In summary, the theory accurately predicted the range of frequencies for which instability can occur, but it underestimated the magnitude of the dynamic load required for

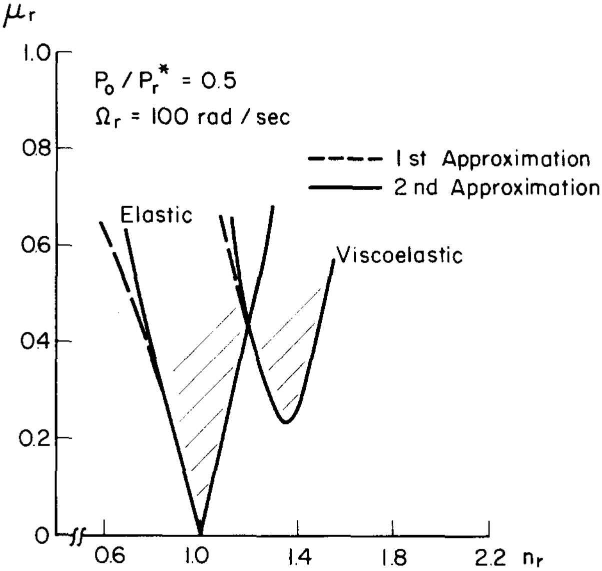

FIG. 3. Influence of viscoelasticity on the principal instability region; cross-hatched area denotes unstable region.

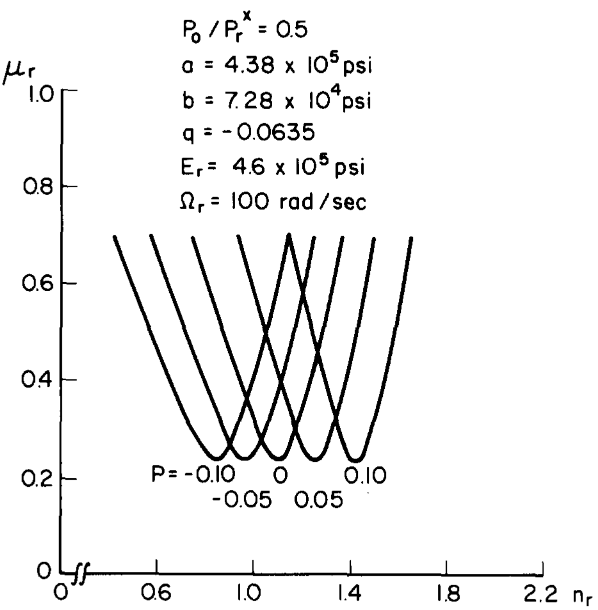

FIG. 4. Effect of the material parameter p on the principal instability region.

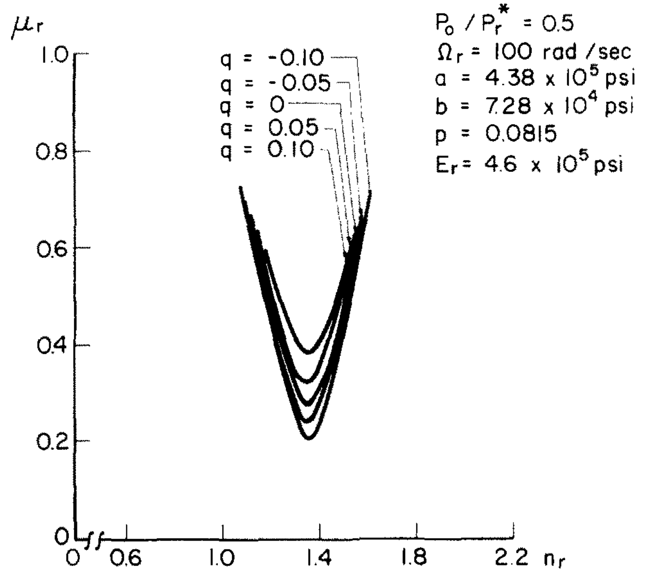

FIG. 5. Effect of the material parameter q on the principal instability region.

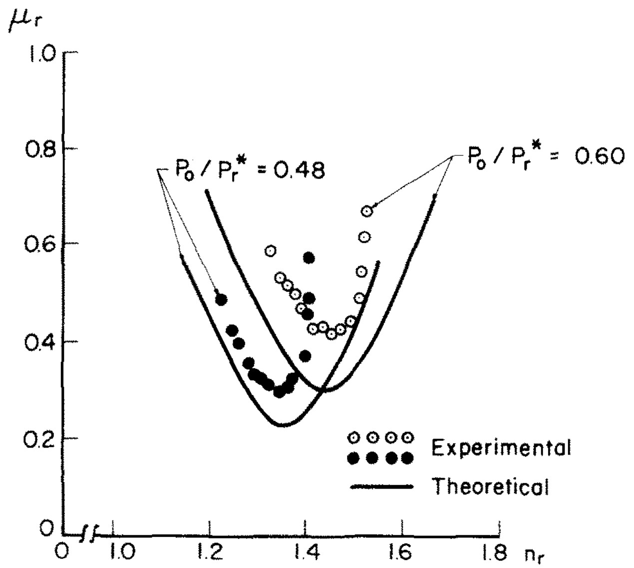

FIG. 6. Principal instability region of a polymethyl methacrylate column.

instability. Both theory and experiment show that the viscoelastic column is stable for rather large dynamic components of load. In this sense, viscoelastic effects tend to stabilize the system. On the other hand, they have a possible destabilizing effect, in that the instability region of the viscoelastic column encompasses a completely different range of frequencies than predicted by an elastic analysis.

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Абстракт—Приводятся результаты аналитических и экспериментальных исследований, касающихся эффекта вязкоупругого поведения материала в зонах неустойчивости колонны, подверженной действию периодической осевой нагрузки $P_0 + P_1$ соз ωt. Для материала используеься комплексное представление модуля. Исследуется эффект характеристик разных типов материала. Оказывается, что вязкоупругие эффекты имеют определенное влияние на зоны неустойчивости. Определяются экспериментально зоны неустойчивости колонны из полиметакрилата метила /плексиглас/. Результаты являются умеренно сходимыми с теорией.